

Sheaf Structures On a Class of Noncommutative Spectra

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Abstract

We introduce a class of noncommutative spectra and give the sheaf structure on the class of noncommutative spectra.

How to select a class of noncommutative rings such that we can rewrite algebraic geometry in the context of the class of noncommutative rings? The purpose of this paper is to present our answer to this interesting problem. It is our belief that if a ring R is in the right class of non-commutative rings which can be used to rewrite algebraic geometry over commutative rings, then R should have the following

Partially Commutative Property: R is a graded ring, and for all homogeneous elements x and y of R , there exist homogeneous elements x' and y' of R such that either $xy = yx'$ or $xy = y'x$.

In this paper, we introduce a class of noncommutative rings which are called Hu-Liu trirings by us. The trivial extension of a ring by a bimodule over the ring has been used in different mathematical areas for a long time ([3] and [6]). Roughly speaking, a Hu-Liu triring is a trivial extension of a ring by a bimodule such that the bimodule carries an extra commutative ring structure. Many results in commutative algebra and algebraic geometry have the satisfactory counterparts in Hu-Liu trirings. The main result of this paper is the sheaf structures on the class of noncommutative spectra based on Hu-Liu trirings. In section 1, we give the basic properties of Hu-Liu trirings. The most important example of Hu-Liu trirings is triquaternions which is defined in section 1. Triquaternions, which are regarded as a kind of new numbers by us, can be used to replace complex numbers to develop the counterpart of complex algebraic geometry. In section 2, we introduce prime triideals and prove some basic facts about prime triideals. In section 3, we use prime triideals to characterize the trinilradical. In section 4, we make the class of noncommutative spectra into a

topological space by introducing extended Zariski topology. In section 5, we define localization of Hu-Liu trirings. In the last section of this paper, we explain how to define the sheaf structures on the class of noncommutative spectra.

Throughout this paper, the word “ring” means an associative ring with an identity. A ring R is also denoted by $(R, +, \cdot)$ to indicate that $+$ is the addition and \cdot is the multiplication in the ring R .

1 Basic Definitions

Let A and B be two subsets of a ring $(R, +, \cdot)$. We shall use $A + B$ and AB to denote the following subsets of R

$$A + B := \{a + b \mid a \in A, b \in B\}, \quad AB := \{ab \mid a \in A, b \in B\}.$$

We now introduce a class of noncommutative rings in the following

Definition 1.1 *A ring R with a multiplication \cdot is called a **Hu-Liu triring** if the following three properties hold.*

- (i) *There exist a commutative subring R_0 of the ring $(R, +, \cdot)$ and a subgroups R_1 of the additive group $(R, +)$, called the **even part** and **odd part** of R respectively, such that $R = R_0 \oplus R_1$ (as Abelian groups) and*

$$R_0 R_0 \subseteq R_0, \quad R_0 R_1 + R_1 R_0 \subseteq R_1, \quad R_1 R_1 = 0; \quad (1)$$

- (ii) *There exists a binary operation \sharp on the odd part R_1 such that $(R_1, +, \sharp)$ is a commutative ring and the two associative products \cdot and \sharp satisfy the **triassociative law**:*

$$x(\alpha \sharp \beta) = (x\alpha) \sharp \beta, \quad (2)$$

$$(\alpha \sharp \beta)x = \alpha \sharp (\beta x), \quad (3)$$

where $x \in R$ and $\alpha, \beta \in R_1$;

- (iii) *For each $x_0 \in R_0$, we have*

$$R_1 x_0 = x_0 R_1. \quad (4)$$

A Hu-Liu triring $R = R_0 \oplus R_1$, which clearly has the partially commutative property, is sometimes denoted by $(R = R_0 \oplus R_1, +, \cdot, \sharp)$, where the associative product \sharp on the odd part R_1 is called the **local product**, and the identity 1^\sharp of the ring $(R_1, +, \sharp)$ is called the **local identity** of the triring R . If $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ is a Hu-Liu triring, then R is the trivial extension of a ring R_0 by a R_0 -bimodule R_1 , and the triassociative law interweaves the

noncommutative ring structure on the trivial extension with the commutative ring structure on the odd part.

Since a commutative ring is a Hu-Liu triring with zero odd part, the concept of Hu-Liu trirings naturally generalizes the concept of commutative rings. The first and the most important example of Hu-Liu trirings which is not a commutative ring is triquaternions whose definition is given in the following example.

Example Let $\mathbf{Q} = \mathcal{R}1 \oplus \mathcal{R}i \oplus \mathcal{R}j \oplus \mathcal{R}k$ be a 4-dimensional real vector space, where \mathcal{R} is the field of real numbers. Then $(\mathbf{Q} = \mathbf{Q}_0 \oplus \mathbf{Q}_1, +, \cdot, \sharp)$ is a Hu-liu triring, where $\mathbf{Q}_0 = \mathcal{R}1 \oplus \mathcal{R}i$ is the even part, $\mathbf{Q}_1 = \mathcal{R}j \oplus \mathcal{R}k$ is the odd part, the ring multiplication \cdot and the local product \sharp are defined by the following multiplication tables:

\cdot	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	0	0
k	k	j	0	0

\sharp	j	k
j	j	k
k	k	$-j$

The Hu-Liu triring \mathbf{Q} is called the **triquaternions**. □

Definition 1.2 Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring with the identity 1, and let I be a subgroup of the additive group of R .

- (i) I is called a **triideal** of R if $IR + RI \subseteq I$, $I = (R_0 \cap I) \oplus (I \cap R_1)$, and $I \cap R_1$ is an ideal of the ring $(R_1, +, \sharp)$.
- (ii) I is called a **subtriring** of R if $1 \in I$, $II \subseteq I$, $I = (R_0 \cap I) \oplus (I \cap R_1)$ and $I \cap R_1$ is a subring of the ring $(R_1, +, \sharp)$.

Let I be a triideal of a Hu-Liu triring $(R = R_0 \oplus R_1, +, \cdot, \sharp)$. It is clear that $\frac{R}{I} = \left(\frac{R}{I}\right)_0 \oplus \left(\frac{R}{I}\right)_1$ with $\left(\frac{R}{I}\right)_i := \frac{R_i + I}{I}$ for $i = 0$ and 1 . We now define a local product on $\left(\frac{R}{I}\right)_1$ by

$$(\alpha + I) \sharp (\beta + I) := \alpha \sharp \beta + I \quad \text{for } \alpha, \beta \in R_1. \quad (5)$$

Then the local product defined by (5) is well-defined, and the triassociative law holds. Therefore, $\frac{R}{I}$ becomes a Hu-Liu triring, which is called the **quotient Hu-Liu triring** of R with respect to the triideal I .

Definition 1.3 Let $R = R_0 \oplus R_1$ and $\overline{R} = \overline{R}_0 \oplus \overline{R}_1$ be Hu-Liu trirings. A map $\phi : R \rightarrow \overline{R}$ is called a **triring homomorphism** if

$$\begin{aligned}\phi(x + y) &= \phi(x) + \phi(y), \quad \phi(xy) = \phi(x)\phi(y), \quad \phi(1_R) = 1_{\overline{R}}, \\ \phi(R_0) &\subseteq \overline{R}_0, \quad \phi(R_1) \subseteq \overline{R}_1, \\ \phi(\alpha \sharp \beta) &= \phi(\alpha) \sharp \phi(\beta), \quad \phi(1^\sharp) = \overline{1}^\sharp\end{aligned}$$

where $x, y \in R$, $\alpha, \beta \in R_1$, 1_R and $1_{\overline{R}}$ are the identities of R and \overline{R} respectively, and 1^\sharp and $\overline{1}^\sharp$ are the local identities of R and \overline{R} respectively. A bijective triring homomorphism is called a **triring isomorphism**. We shall use $R \simeq \overline{R}$ to indicate that there exists a triring isomorphism from R to \overline{R} .

Let ϕ be a triring homomorphism from a Hu-Liu triring R to a Hu-Liu triring \overline{R} . the **kernel** $\text{Ker}\phi$ and the **image** $\text{Im}\phi$ of ϕ are defined by

$$\text{Ker}\phi := \{x \mid a \in R \text{ and } \phi(x) = 0\}$$

and

$$\text{Im}\phi := \{\phi(x) \mid x \in R\}.$$

Clearly, $\text{Ker}\phi$ is a triideal of R , $\text{Im}\phi$ is a subtriring of \overline{R} and

$$\overline{\phi} : x + \text{Ker}\phi \rightarrow \phi(x) \quad \text{for } x \in R$$

is a triring isomorphism from the quotient Hu-Liu triring $\frac{R}{\text{Ker}\phi}$ to the subtriring $\text{Im}\phi$ of \overline{R} . If I is triideal of R , then

$$\nu : x \mapsto x + I \quad \text{for } x \in R$$

is a surjective triring homomorphism from R the quotient triring $\frac{R}{I}$ with kernel I . The map ν is called the **natural triring homomorphism**.

Proposition 1.1 Let ϕ be a surjective homomorphism from a Hu-Liu triring $R = R_0 \oplus R_1$ to a Hu-Liu triring $\overline{R} = \overline{R}_0 \oplus \overline{R}_1$.

(i) $\phi(R_i) = \overline{R}_i$ for $i = 0$ and 1 .

(ii) Let

$$\mathcal{S} := \{I \mid I \text{ is a triideal of } R \text{ and } I \supseteq \text{Ker}\phi\}$$

and

$$\overline{\mathcal{S}} := \{\overline{I} \mid \overline{I} \text{ is a triideal of } \overline{R}\}.$$

The map

$$\Psi : I \mapsto \phi(I) := \{\phi(x) \mid x \in I\}$$

is a bijection from \mathcal{S} to $\overline{\mathcal{S}}$, and the inverse map $\Psi^{-1} : \overline{\mathcal{S}} \rightarrow \mathcal{S}$ is given by

$$\Psi^{-1} : \overline{I} \mapsto \phi^{-1}(\overline{I}) := \{x \mid x \in R \text{ and } \phi(x) \in \overline{I}\}.$$

(iii) If I is a triideal of R containing $\text{Ker}\phi$, then the map

$$x + I \mapsto \phi(x) + \phi(I) \quad \text{for } x \in R$$

is a triring isomorphism from the quotient triring $\frac{R}{I}$ onto the quotient triring $\frac{\overline{R}}{\phi(I)}$.

Proof A routine check. □

We now prove a basic property for Hu-Liu trirings.

Proposition 1.2 Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring. If $x_i \in R_i$ for $i \in \{0, 1\}$, then both $Rx_0 = R_0x_0 \oplus R_1x_0$ and $R_1\#x_1$ are triideals of R .

Proof Since $(R_1, +, \#)$ is a commutative ring, $R_1\#x_1$ is clearly a triideal of R .

Using the triassociative law and (4), we have

$$(R_i(R_0x_0 + R_1x_0)) \cup (R_0x_0 + R_1x_0)R_i \subseteq R_0x_0 + R_1x_0 \quad (6)$$

for $i = 0, 1$. By (6), $R_0x_0 \oplus R_1x_0$ is an ideal of the ring $(R, +, \cdot)$. Also, $R_1x_0 = R_1\#(1\#x_0)$ is obviously an ideal of the commutative ring $(R_1, +, \#)$. Thus, $R_0x_0 \oplus R_1x_0$ is a triideal of R . □

The next position gives some operations about triideals in a Hu-Liu triring.

Proposition 1.3 Let I, J, I_λ with $\lambda \in \Lambda$ be triideals of a Hu-Liu triring R .

(i) The intersection $I \cap J$ and the sum $\sum_{\lambda \in \Lambda} I_\lambda$ are triideals of R . Moreover, we

$$\text{have } (I \cap J)_i = I_i \cap J_i \text{ and } \left(\sum_{\lambda \in \Lambda} I_\lambda \right)_i = \sum_{\lambda \in \Lambda} (I_\lambda)_i \text{ for } i = 0, 1.$$

(ii) The **mixed product** $I \dot{\#} J := (I \dot{\#} J)_0 \oplus (I \dot{\#} J)_1$ of I and J is a triideal, where $(I \dot{\#} J)_0 = I_0J_0$ and $(I \dot{\#} J)_1 = I_1\#J_1$.

Proof (i) It is clear.

(ii) By triassociative law, we have

$$R_0(I \dot{\#} J) \subseteq R_0I_0J_0 + R_0(I_1\#J_1) \subseteq I_0J_0 + (R_0I_1)\#J_1 \subseteq I \dot{\#} J,$$

$$(I \dot{\#} J)R_0 \subseteq I_0J_0R_0 + (I_1\#J_1)R_0 \subseteq I_0J_0 + I_1\#(J_1R_0) \subseteq I \dot{\#} J,$$

$$R_1(I \dot{\#} J) + (I \dot{\#} J)R_1 \subseteq R_1I_0J_0 + I_0J_0R_1 \subseteq I_1J_0 + I_0J_1 \subseteq I \dot{\#} J$$

and $R_1\#(I_1\#J_1) \subseteq R_1\#I_1\#J_1 \subseteq I \dot{\#} J$. This proves that (ii) holds. □

2 Prime Triideals

We begin this section by introducing the notion of prime triideals.

Definition 2.1 Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring. An triideal $P = P_0 \oplus P_1$ of R is called a **prime triideal** if $P \neq R$ and

$$x_0 y_0 \in P_0 \Rightarrow x_0 \in P_0 \text{ or } y_0 \in P_0, \quad (7)$$

$$x_0 y_1 \in P_1 \Rightarrow x_0 \in P_0 \text{ or } y_1 \in P_1, \quad (8)$$

$$x_1 y_0 \in P_1 \Rightarrow x_1 \in P_1 \text{ or } y_0 \in P_0, \quad (9)$$

$$x_1 \# y_1 \in P_1 \Rightarrow x_1 \in P_1 \text{ or } y_1 \in P_1, \quad (10)$$

where $x_i, y_i \in R_i$ for $i = 0$ and 1 .

Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring. The set of all prime triideals of R is called the **trispectrum** of R and denoted by $\text{Spec}^\# R$. It is clear that

$$\text{Spec}^\# R = \text{Spec}_0^\# R \cup \text{Spec}_1^\# R \quad \text{and} \quad \text{Spec}_0^\# R \cap \text{Spec}_1^\# R = \emptyset,$$

where

$$\text{Spec}_0^\# R := \{ P \mid P \in \text{Spec}^\# R \text{ and } P \supseteq R_1 \}$$

is called the **even trispectrum** of R and

$$\text{Spec}_1^\# R := \{ P \mid P \in \text{Spec}^\# R \text{ and } P \not\supseteq R_1 \}$$

is called the **odd trispectrum** of R .

Let $P = P_0 \oplus P_1$ be a triideal of a Hu-Liu triring R . Then $P \in \text{Spec}_0^\# R$ if and only if $P_1 = R_1$ and P_0 is a prime ideal of the commutative ring $(R_0, +, \cdot)$. It is also obvious that $P \in \text{Spec}_1^\# R$ if and only if P_0 is a prime ideal of the commutative ring $(R_0, +, \cdot)$, P_1 is a prime ideal of the commutative ring $(R_1, +, \#)$, and the $\left(\frac{R_0}{P_0}, \frac{R_0}{P_0}\right)$ -bimodule $\frac{R}{P}$ is faithful as both left module and right module, where the left $\frac{R_0}{P_0}$ -module action on $\frac{R}{P}$ is defined by

$$(x_0 + P_0)(y + P) := x_0 y + P \quad \text{for } x_0 \in P_0 \text{ and } y \in R$$

and the right $\frac{R_0}{P_0}$ -module action on $\frac{R}{P}$ is defined by

$$(y + P)(x_0 + P_0) := y x_0 + P \quad \text{for } x_0 \in P_0 \text{ and } y \in R.$$

Clearly, the even trispectrum $\text{Spec}_0^\# R$ of a Hu-Liu triring R is not empty. A basic property of Hu-Liu trirings is that the odd trispectrum $\text{Spec}_1^\# R$ is always not empty provided $R_1 \neq 0$. This basic fact is a corollary of the following

Proposition 2.1 *Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring with $R_1 \neq 0$. If P_1 is a prime ideal of the commutative ring $(R_1, +, \#)$, then there exists prime ideal P_0 of the commutative ring $(R_0, +, \cdot)$ such that $P := P_0 \oplus P_1$ is a prime triideal of R , and P_0 contains every ideal I_0 of the ring $(R_0, +, \cdot)$ which has the property: $R_1 I_0 \subseteq P_1$.*

Proof Consider the set Ω defined by

$$\Omega := \{ I_0 \mid I_0 \text{ is an ideal of } (R_0, +, \cdot) \text{ and } R_1 I_0 \subseteq P_1 \}.$$

Clearly, $1 \notin I_0$ if $I_0 \in \Omega$. Since $0 \in \Omega$, Ω is not empty. The relation of inclusion, \subseteq , is a partial order on Ω . Let Δ be a non-empty totally ordered subset of Ω . Let $J_0 := \bigcup_{I_0 \in \Delta} I_0$. Then $J_0 \in \Omega$. Thus J_0 is an upper bound for

Δ in Ω . By Zorn's Lemma, the partial order set (Ω, \subseteq) has a maximal element P_0 . We are going to prove that $P := P_0 \oplus P_1$ is a prime triideal of R . Clearly, $P = P_0 \oplus P_1$ is a triideal satisfying (10). Let $x_i, y_i \in R_i$ for $i = 0, 1$.

If $x_1 y_0 \in P_1$ and $x_1 \notin P_1$, then $x_1 \# (1^\# y_0) = x_1 y_0 \in P_1$, which implies that $1^\# y_0 \in P_1$. Hence, we have

$$\begin{aligned} R_1(P_0 + R_0 y_0) &\subseteq R_1 P_0 + R_1 R_0 y_0 \subseteq P_1 + R_1 y_0 \\ &= P_1 + R_1 \# (1^\# y_0) \subseteq P_1 + R_1 \# P_1 \subseteq P_1. \end{aligned} \quad (11)$$

Using (11) and the fact that $P_0 + R_0 y_0$ is an ideal of R_0 , we get $P_0 + R_0 y_0 \in \Omega$. Since $P_0 + R_0 y_0 \supseteq P_0$, we have to have $P_0 + R_0 y_0 = P_0$, which implies that $y_0 \in P_0$. This proves that (9) holds.

If $x_0 y_1 \in P_1$ and $y_1 \notin P_1$, then $(x_0 1^\#) \# y_1 = x_0 y_1 \in P_1$, which implies that $x_0 1^\# \in P_1$. It follows from this fact and (4) that

$$\begin{aligned} R_1(P_0 + R_0 x_0) &\subseteq R_1 P_0 + R_1 R_0 x_0 \subseteq P_1 + R_1 x_0 = P_1 + x_0 R_1 \\ &= P_1 + (x_0 1^\#) \# R_1 \subseteq P_1 + P_1 \# R_1 \subseteq P_1. \end{aligned} \quad (12)$$

Using (12) and the fact that $P_0 + R_0 x_0$ is an ideal of R_0 , we get $P_0 + R_0 x_0 \in \Omega$. Since $P_0 + R_0 x_0 \supseteq P_0$, we have to have $P_0 + R_0 x_0 = P_0$, which implies that $x_0 \in P_0$. This proves that (8) holds.

Finally, if $x_0 y_0 \in P_0$, then $(1^\# x_0) \# (1^\# y_0) = 1^\# x_0 y_0 \in P_1$. Thus, $(1^\# x_0) \in P_1$ or $(1^\# y_0) \in P_1$, which implies that either $x_0 \in P_0$ or $y_0 \in P_0$ by the argument above. This proves that (7) holds.

Summarizing what we have proved, $P = P_0 \oplus P_1$ is a prime triideal of R . \square

The following proposition gives another characterization of prime triideals.

Proposition 2.2 *Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring. The following are equivalent.*

(i) P is a prime triideal.

(ii) For two triideals I, J of R , $I \# J \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$.

Proof (i) \Rightarrow (ii): Assume that $I \# J \subseteq P$ and $I \not\subseteq P$. Then either $I_0 \not\subseteq P_0$ or $I_1 \not\subseteq P_1$. Let $y_i \in J_i$ with $i = 0, 1$.

If $I_0 \not\subseteq P_0$, then there exists $x_0 \in P_0 \setminus I_0$. Clearly, $x_0 y_i \in I \# J \subseteq P$ for $i = 0, 1$. Since P is a prime triideal, we get $y_i \in P$ for $i = 0, 1$ by (7) and (8). Thus $J \subseteq P$ in this case.

If $I_1 \not\subseteq P_1$, then there exists $x_1 \in P_1 \setminus I_1$. Since $x_1 y_0 \in I \# J \subseteq P$ and $x_1 \# y_1 \in I \# J \subseteq P$, we get $y_i \in P$ for $i = 0, 1$ by (9) and (10). Thus we also get $J \subseteq P$ in this case.

(ii) \Rightarrow (i): Let $x_i, y_i \in P_i$ with for $i = 0, 1$. By Proposition 1.2, both $R_0 x_0 \oplus R_1 x_0$ and $R_0 y_0 \oplus R_1 y_0$ are triideals. By the definition of the mixed product of two triideals in Proposition 1.3, we have

$$(R_0 x_0 \oplus R_1 x_0) \# (R_0 y_0 \oplus R_1 y_0) = R_0 x_0 R_0 y_0 + (R_1 x_0) \# (R_1 y_0).$$

Since $(R_1 x_0) \# (R_1 y_0) = R_1 x_0 y_0$, we get

$$(R_0 x_0 \oplus R_1 x_0) \# (R_0 y_0 \oplus R_1 y_0) \subseteq R_0 x_0 y_0 + R_1 x_0 y_0. \quad (13)$$

If $x_0 y_0 \in P_0$ and $x_0 \notin P_0$, then

$$R_0 x_0 \oplus R_1 x_0 \not\subseteq P. \quad (14)$$

It follows from (13) and (14) that $R_0 y_0 \oplus R_1 y_0 \subseteq P$, which implies that $y_0 \in P_0$. Thus, (7) holds.

If $x_0 y_1 \in P_1$ and $x_0 \notin P_0$, then (14) holds. By (4), $1^\# x_0 = x_0 z_1$ for some $z_1 \in R_1$. Hence, we have

$$\begin{aligned} (R_0 x_0 \oplus R_1 x_0) \# (R_1 \# y_1) &= (R_1 x_0) \# (R_1 \# y_1) \subseteq R_1 \# (1^\# x_0) \# y_1 \\ &\subseteq R_1 \# (x_0 z_1) \# y_1 \subseteq R_1 \# (x_0 1^\#) \# z_1 \# y_1 \subseteq R_1 \# z_1 \# (x_0 1^\#) \# y_1 \\ &\subseteq R_1 \# (x_0 (1^\# \# y_1)) \subseteq R_1 \# (x_0 y_1). \end{aligned} \quad (15)$$

It follows from (13) and (15) that $R_1 \# y_1 \subseteq P$, which implies that $y_1 \in P$. Thus, (8) holds.

If $x_1 y_0 \in P_1$ and $x_1 \notin P_1$, then

$$R_1 \# x_1 \not\subseteq P. \quad (16)$$

It follows from $x_1 y_0 \in P_1$ that

$$\begin{aligned} (R_1 \# x_1) \# (R_0 y_0 + R_1 y_0) &= (R_1 \# x_1) \# (R_1 y_0) = R_1 \# x_1 \# R_1 \# (1^\# y_0) \\ &= R_1 \# x_1 \# (1^\# y_0) = R_1 \# (x_1 \# 1^\#) y_0 = R_1 \# (x_1 y_0) \subseteq P. \end{aligned} \quad (17)$$

By (16) and (17), we get $R_0 y_0 + R_1 y_0 \subseteq P$, which implies that $y_0 \in P$. Thus, (9) holds.

If $x_1 \# y_1 \in P_1$ and $x_1 \notin P_1$, then (16) holds and

$$(R_1 \# x_1) \# (R_1 \# y_1) = R_1 \# x_1 \# y_1 \subseteq P. \quad (18)$$

By (16) and (18), $R_1 \# y_1 \subseteq P$, which implies that $y_1 \in P$. Thus, (10) holds.

This proves that P is a prime triideal. \square

3 Trinilradicals

Let $R = R_0 \oplus R_1$ be a Hu-Liu triring with the local identity $1^\#$. For $\alpha \in R_1$, the **local n th power** $\alpha^{\#n}$ is defined by:

$$\alpha^{\#n} := \begin{cases} 1^\#, & \text{if } n = 0; \\ \underbrace{\alpha \# \alpha \# \cdots \# \alpha}_n, & \text{if } n \text{ is a positive integer.} \end{cases}$$

The products $(x^m)(\alpha^{\#n})$ and $(\alpha^{\#n})(x^m)$ will be denoted by $x^m \alpha^{\#n}$ and $\alpha^{\#n} x^m$ respectively, where $x \in R$ and $\alpha \in R_1$.

Proposition 3.1 *Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring with the local identity $1^\#$. If $x, y \in R$, $\alpha, \beta \in R_1$ and $m \in \mathbb{Z}_{>0}$, then*

$$(x\alpha) \# (y\beta) = (xy)(\alpha \# \beta), \quad (\alpha x) \# (\beta y) = (\alpha \# \beta)xy \quad (19)$$

and

$$(x\alpha)^{\#m} = x^m \alpha^{\#m}, \quad (\alpha x)^{\#m} = \alpha^{\#m} x^m. \quad (20)$$

Proof By the triassociative law, we have

$$(x\alpha) \# (y\beta) = x(\alpha \# (y\beta)) = x((y\beta) \# \alpha) = (xy)(\beta \# \alpha) = (xy)(\alpha \# \beta)$$

and

$$(\alpha x) \# (\beta y) = ((\alpha x) \# \beta) y = ((\beta) \# (\alpha x)) y = (\beta \# \alpha)xy = (\alpha \# \beta)xy.$$

Hence, (19) holds. Clearly, (20) follows from (19). \square

Definition 3.1 Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring.

(i) An element x of R is said to be **trinilpotent** if

$$x_0^m = 0 \text{ and } x_1^{\sharp n} = 0 \text{ for some } m, n \in \mathbb{Z}_{>0}, \quad (21)$$

where x_0 and x_1 are the even component and the odd component of x respectively.

(ii) The set of all graded nilpotent elements of R is called the **trinilradical** of R and denoted by $\text{nilrad}^\sharp(R)$ or $\sqrt[3]{0}$.

The ordinary nilradical of a ring $(A, +, \cdot)$ is denoted by $\text{nilrad}(A)$ or $\text{nilrad}(A, +, \cdot)$; that is,

$$\text{nilrad}(A) := \{x \mid x^m = 0 \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

If $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ is a Hu-Liu triring, then

$$\text{nilrad}(R, +, \cdot) = \text{nilrad}(R_0, +, \cdot) \oplus R_1$$

and

$$\text{nilrad}^\sharp R = \text{nilrad}(R_0, +, \cdot) \oplus \text{nilrad}(R_1, +, \sharp). \quad (22)$$

Hence, $\text{nilrad}^\sharp R \subseteq \text{nilrad}(R, +, \cdot)$, i.e., the trinilradical of a Hu-Liu triring R is smaller than the ordinary nilradical of the ring $(R, +, \cdot)$.

Proposition 3.2 Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring.

(i) The trinilradical $\text{nilrad}^\sharp(R)$ is a triideal of R .

(ii) $\text{nilrad}^\sharp\left(\frac{R}{\text{nilrad}^\sharp(R)}\right) = 0$.

Proof (i) By the definition of trinilradicals, we have

$$(\text{nilrad}^\sharp R) \cap R_0 = \text{nilrad}(R_0, +, \cdot) \quad \text{and} \quad (\text{nilrad}^\sharp R) \cap R_1 = \text{nilrad}(R_1, +, \sharp).$$

Using the fact above and (22), we need only to prove

$$\{xa, ax\} \subseteq \text{nilrad}^\sharp R \quad \text{for } x \in R \text{ and } a \in \text{nilrad}^\sharp R. \quad (23)$$

Let $x = x_0 + x_1 \in R$ and $a = a_0 + a_1 \in \text{nilrad}^\sharp R$, where $x_i, a_i \in R_i$ for $i = 0, 1$. Then we have $a_0^m = a_1^{\sharp m} = 0$ for some $m \in \mathbb{Z}_{>0}$. Since

$$\begin{aligned} xa &= (x_0 + x_1)(a_0 + a_1) = x_0a_0 + (x_0a_1 + x_1a_0), \\ (x_0a_0)^m &= x_0^m a_0^m = x_0^m 0 = 0, \\ (x_0a_1)^{\sharp m} &= x_0^m a_1^{\sharp m} = x_0^m 0 = 0, \\ (x_1a_0)^{\sharp m} &= (x_1a_0)^{\sharp m} = x_1^{\sharp m} a_0^m = x_1^{\sharp m} 0 = 0, \end{aligned} \quad (24)$$

we get

$$x_0 a_0 \in \text{nilrad}(R_0, +, \cdot) \quad \text{and} \quad x_0 a_1 + x_1 a_0 \in \text{nilrad}^\sharp(R_1, +, \sharp). \quad (25)$$

It follows from (24) and (25) that $xa \in \text{nilrad}^\sharp R$. Similarly, we have $ax \in \text{nilrad}^\sharp R$. This proves (i).

(ii) If $x + \text{nilrad}^\sharp R \in \frac{R}{\text{nilrad}^\sharp R}$, then there exist positive integers m and n such that

$$x_0^m + \text{nilrad}^\sharp R = (x_0 + \text{nilrad}^\sharp R)^m = \text{nilrad}^\sharp R$$

and

$$x_1^{\sharp n} + \text{nilrad}^\sharp R = ((x_1 + \text{nilrad}^\sharp R)_1)^{\sharp n} = \text{nilrad}^\sharp R,$$

where $x = x_0 + x_1$ and $x_i \in R_i$ for $i = 0, 1$. Hence, we get

$$x_0^m \in \text{nilrad}^\sharp R \quad \text{and} \quad x_1^{\sharp n} \in \text{nilrad}^\sharp R,$$

which imply that

$$x_0^{mu} = (x_0^m)^u = 0 \quad \text{and} \quad x_1^{\sharp(nv)} = (x_1^{\sharp n})^{\sharp v} = 0$$

for some $u, v \in \mathbb{Z}_{>0}$. Thus, $x = x_0 + x_1 \in \text{nilrad}^\sharp R$ or $x + \text{nilrad}^\sharp R$ is the zero element of $\frac{R}{\text{nilrad}^\sharp R}$. This proves (ii). \square

If I is a triideal of a Hu-Liu triring $R = R_0 \oplus R_1$, then the **trinilradical** $\sqrt[\sharp]{I}$ of I is defined by

$$\sqrt[\sharp]{I} := \{ x \in R \mid x_0^m \in I_0 \text{ and } x_1^{\sharp n} \in I_1 \text{ for some } m, n \in \mathbb{Z}_{>0} \},$$

where $x = x_0 + x_1$, $x_i \in R_i$ and $I_i = I \cap R_i$ for $i = 0, 1$. Since

$$\text{nilrad}^\sharp \left(\frac{R}{I} \right) = \frac{\sqrt[\sharp]{I}}{I},$$

$\sqrt[\sharp]{I}$ is a triideal of R . A triideal I of a Hu-Liu triring R is called a **radical triideal** if $\sqrt[\sharp]{I} = I$.

We now characterize the trinilradical of a Hu-Liu triring by using prime triideals.

Proposition 3.3 *Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring. The trinilradical of R is the intersection of the prime triideals of R .*

Proof Let $x = x_0 + x_1$ be any element of $\text{nilrad}^\sharp(R)$, where $x_0 \in R_0$ and $x_1 \in R_1$. Then $x_0^m = 0$ and $x_1^{\sharp n} = 0$ for some $m, n \in \mathbb{Z}_{>0}$. Let $P = P_0 \oplus P_1$ be any prime triideal of R . Since $x_0^m = 0 \in P_0$, we have

$$x_0 \in P_0 \quad (26)$$

by (7).

If $P \not\supseteq R_1$, then P_1 is a prime ideal of the commutative ring $(R_1, +, \#)$. Using this fact and $x_1^{\#n} = 0$, we get

$$x_1 \in P_1. \quad (27)$$

If $P \supseteq R_1$, then (27) is obviously true. It follows from (26) and (27) that $x = x_0 + x_1 \in P$. This proves that

$$\text{nilrad}^\#(R) \subseteq \bigcap_{P \in \text{Spec}^\# R} P. \quad (28)$$

Conversely, we prove that

$$z \notin \text{nilrad}^\#(R) \Rightarrow z \notin \bigcap_{P \in \text{Spec}^\# R} P. \quad (29)$$

Case 1: $z^m \neq 0$ for all $m \in \mathbb{Z}_{>0}$, in which case, $z^m \notin R_1$ for all $m \in \mathbb{Z}_{>0}$. Hence, $z + R_1$ is not a nilpotent element of the commutative ring $\frac{R}{R_1}$; that is,

$$z + R_1 \neq \text{nilrad}\left(\frac{R}{R_1}\right) = \bigcap_{\frac{I}{R_1} \in \text{Spec}\left(\frac{R}{R_1}\right)} \left(\frac{I}{R_1}\right),$$

where $\text{Spec}\left(\frac{R}{R_1}\right)$ is the ordinary spectrum of the commutative ring $\left(\frac{R}{R_1}, +, \cdot\right)$.

Hence, there exists a prime ideal $\frac{I}{R_1}$ of the commutative ring $\frac{R}{R_1}$ such that $z \notin I$. Since I is a prime triideal of R , (29) holds in this case.

Case 2: $z^m = 0$ for some $m \in \mathbb{Z}_{>0}$. Let $z = z_0 + z_1$ with $z_0 \in R_0$ and $z_1 \in R_1$. Then $0 = z^m = z_0^m + z_0^{m-1}z_1$ for some $r_1 \in R_1$ by (4). Thus, $z_0^m = 0$, which implies that $z_1^{\#n} \neq 0$ for all $n \in \mathbb{Z}_{>0}$ in this case. We now consider the following set

$$T := \left\{ J \mid J \text{ is a triideal of } R \text{ and } z_1^{\#n} \notin J \text{ for all } n \in \mathbb{Z}_{>0} \right\}.$$

Since $\{0\} \in T$, T is nonempty. Clearly, (T, \subseteq) is a partially order set, where \subseteq is the relation of set inclusion. If $\{J_\lambda \mid \lambda \in \Lambda\}$ is a nonempty totally ordered subset of T , then $\bigcup_{\lambda \in \Lambda} J_\lambda$ is an upper bound of $\{J_\lambda \mid \lambda \in \Lambda\}$ in T . By Zorn's Lemma, the partially ordered set (T, \subseteq) has a maximal element P . We are going to prove that P is a prime triideal of R .

Let $x = x_0 + x_1$ and $y = y_0 + y_1$ be two elements of R , where $x_i, y_i \in R_i$ for $i = 0, 1$. First, if $x_0 \notin P_0$ and $y_0 \notin P_0$, then

$$P \subset P + Rx_0 \quad \text{and} \quad P \subset P + Ry_0. \quad (30)$$

Since both $P + Rx_0$ and $P + Ry_0$ are triideals of R by Proposition 1.2, (30) implies that

$$z_1^{\#u} \in P + Rx_0 \quad \text{and} \quad z_1^{\#v} \in P + Ry_0$$

or

$$z_1^{\#u} \in P_1 + R_1x_0 \quad \text{and} \quad z_1^{\#v} \in P_1 + R_1y_0 \quad \text{for some } u, v \in \mathcal{Z}_{>0}.$$

Thus, we have

$$\begin{aligned} z_1^{\#(u+v)} &= z_1^{\#u} \# z_1^{\#v} \in (P_1 + R_1x_0) \# (P_1 + R_1y_0) \\ &\subseteq \underbrace{P_1 \# P_1 + P_1 \# (R_1y_0) + (R_1x_0) \# P_1 + (R_1x_0) \# (R_1y_0)}_{\text{This is a subset of } P} \\ &\subseteq P + R_1x_0y_0, \end{aligned}$$

which implies that $x_0y_0 \notin P$. This proves that

$$x_0 \notin P_0 \text{ and } y_0 \notin P_0 \Rightarrow x_0y_0 \notin P_0. \quad (31)$$

Next, if $x_0 \notin P_0$ and $y_1 \notin P_1$, then

$$P \subset P + Rx_0 \quad \text{and} \quad P \subset P + R_1 \# y_1. \quad (32)$$

Since both $P + Rx_0$ and $P + R_1 \# y_1$ are triideals of R by Proposition 1.2, (32) implies that there exist some positive integers s and t such that

$$z_1^{\#s} \in P + Rx_0 \quad \text{and} \quad z_1^{\#t} \in P + R_1 \# y_1$$

or

$$z_1^{\#s} \in P_1 + R_1x_0 \quad \text{and} \quad z_1^{\#t} \in P_1 + R_1 \# y_1. \quad (33)$$

It follows that

$$\begin{aligned} z_1^{\#(s+t)} &= z_1^{\#s} \# z_1^{\#t} \in (P_1 + R_1x_0) \# (P_1 + R_1 \# y_1) \\ &\subseteq \underbrace{P_1 \# P_1 + P_1 \# R_1 \# y_1 + (R_1x_0) \# P_1 + (R_1x_0) \# R_1 \# y_1}_{\text{This is a subset of } P_1} \\ &\subseteq P_1 + R_1 \# (x_0y_1), \end{aligned}$$

which implies that $x_0y_1 \notin P_1$. This proves that

$$x_0 \notin P_0 \text{ and } y_1 \notin P_1 \Rightarrow x_0y_1 \notin P_1. \quad (34)$$

Similarly, we have

$$y_1 \notin P_1 \text{ and } x_0 \notin P_0 \Rightarrow y_1x_0 \notin P_1 \quad (35)$$

and

$$x_1 \notin P_1 \text{ and } y_1 \notin P_1 \Rightarrow x_1 y_1 \notin P_0. \quad (36)$$

By (31), (34), (35) and (36), P is a prime triideal. Since $z_1 \notin P$, (29) also holds in Case 2.

It follows from (28) and (29) that Proposition 3.3 is true. \square

The next proposition is a corollary of Proposition 3.3.

Proposition 3.4 *If I is a triideal of a Hu-Liu triring R and $I \neq R$, then*

$$\sqrt[\sharp]{I} = \bigcap_{P \in \text{Spec}^\sharp R \text{ and } P \supseteq I} P.$$

Proof By Proposition 3.3, we have

$$\begin{aligned} x \in \sqrt[\sharp]{I} &\Leftrightarrow x + I \in \text{nilrad}^\sharp\left(\frac{R}{I}\right) = \bigcap_{\frac{P}{I} \in \text{Spec}^\sharp\left(\frac{R}{I}\right)} \frac{P}{I} \\ &\Leftrightarrow x \in \bigcap_{P \in \text{Spec}^\sharp R \text{ and } P \supseteq I} P. \end{aligned}$$

\square

4 Extended Zariski Topology

Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring. For a triideal I of R , we define a subset $\mathcal{V}(I)$ of $\text{Spec}^\sharp R$ by

$$V^\sharp(I) := \{P \mid P \in \text{Spec}^\sharp R \text{ and } P \supseteq I\}. \quad (37)$$

Proposition 4.1 *Let R be a Hu-Liu triring.*

- (i) $V^\sharp(0) = \text{spec}^\sharp R$ and $V^\sharp(R) = \emptyset$.
- (ii) $V^\sharp(I) \cup V^\sharp(J) = V^\sharp(I \cap J) = V^\sharp(I \sharp J)$, where I and J are two triideals of R .
- (iii) $\bigcap_{\substack{\lambda \in \Lambda \\ \text{of } R}} V^\sharp(I_{(\lambda)}) = V^\sharp\left(\sum_{\lambda \in \Lambda} I_{(\lambda)}\right)$, where $\{I_{(\lambda)} \mid \lambda \in \Lambda\}$ is a set of triideals of R .

Proof Since (i) and (iii) are clear, we need only to prove (ii).

Since $I \dot{\#} J \subseteq I \cap J \subseteq I$, we get $V^\#(I \dot{\#} J) \supseteq V^\#(I \cap J) \supseteq V^\#(I)$. Similarly, we have $V^\#(I \dot{\#} J) \supseteq V^\#(I \cap J) \supseteq V^\#(J)$. Thus, we get

$$V^\#(I) \cup V^\#(J) \subseteq V^\#(I \cap J) \subseteq V^\#(I \dot{\#} J). \quad (38)$$

Conversely, if $P \in V^\#(I \dot{\#} J)$, then $I \dot{\#} J \subseteq P$. By Proposition 2.2, we get $I \subseteq P$ or $J \subseteq P$. Hence, $P \in V^\#(I) \cup V^\#(J)$. This proves that

$$V^\#(I \dot{\#} J) \subseteq V^\#(I) \cup V^\#(J). \quad (39)$$

It follows from (38) and (39) that (ii) is true. \square

Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring. By Proposition 4.1, the collection

$$V^\# := \{ V^\#(I) \mid I \text{ is a triideal of } R \}$$

of subsets of $\text{Spec}^\# R$ satisfies the axioms for closed sets in a topological space. The topology on $\text{Spec}^\# R$ having the elements of $V^\#$ as closed sets is called the **extended Zariski topology**. The collection

$$D^\# := \{ D^\#(I) \mid I \text{ is a triideal of } R \}$$

consists of the open sets of the extended Zariski topology on $\text{Spec}^\# R$, where

$$D^\#(I) := \text{Spec}^\# R \setminus V^\#(I) = \{ P \mid P \in \text{Spec}^\# R \text{ and } P \not\supseteq I \}.$$

For $x_i \in R_i$ with $i \in \{0, 1\}$, both Rx_0 and $R_1 \# x_1$ are triideals of R by Proposition 1.2. Let

$$D^\#(x_0) := D^\#(Rx_0), \quad D^\#(x_1) := D^\#(R_1 \# x_1).$$

If I_0 and I_1 are the even part and odd part of an triideal I , then

$$D^\#(I) = \bigcup_{x_i \in I_i, i=0,1} D^\#(x_i).$$

Thus, $\{ D^\#(x_i) \mid x_i \in R_i \text{ with } i = 0, 1 \}$ forms an open base for the extended Zariski topology on $\text{Spec}^\# R$. Each $D^\#(x_i)$ is called a **basic open subset** of $\text{Spec}^\# R$. Clearly, $D^\#(0) = \emptyset$, $D^\#(1) = \text{Spec}^\# R$, $D^\#(1^\#) = \text{Spec}_1^\# R$, and $D^\#(x_1) \subseteq \text{Spec}_1^\# R$ for $x_1 \in R_1$.

Proposition 4.2 *Let I and J be triideals of a Hu-Liu triring.*

(i) $V^\sharp(I) \subseteq V^\sharp(J)$ if and only if $\sqrt[\sharp]{J} \subseteq \sqrt[\sharp]{I}$.

(ii) $V^\sharp(I) = V^\sharp(\sqrt[\sharp]{I})$.

Proof (i) If $V^\sharp(I) \subseteq V^\sharp(J)$, then $P \supseteq I$ implies that $P \supseteq J$ for $P \in \text{Spec}^\sharp R$. Thus, we have

$$\begin{aligned} & \{P \mid P \in \text{Spec}^\sharp R \text{ and } P \supseteq J\} \\ = & \{P \mid P \in \text{Spec}^\sharp R \text{ and } P \supseteq I\} \bigcup \{P \mid P \in \text{Spec}^\sharp R, P \supseteq J \text{ and } P \not\supseteq I\}. \end{aligned}$$

By Proposition 3.4, we get

$$\begin{aligned} \sqrt[\sharp]{J} &= \bigcap_{\substack{P \in \text{spec}^\sharp R \\ P \supseteq J}} P = \left(\bigcap_{\substack{P \in \text{spec}^\sharp R \\ P \supseteq I}} P \right) \cap \left(\bigcap_{\substack{P \in \text{spec}^\sharp R \\ P \supseteq J, P \not\supseteq I}} P \right) \\ &\subseteq \bigcap_{\substack{P \in \text{spec}^\sharp R \\ P \supseteq I}} P = \sqrt[\sharp]{J}. \end{aligned}$$

Conversely, if $\sqrt[\sharp]{J} \subseteq \sqrt[\sharp]{I}$, then for any $Q \in V^\sharp(I)$, we have $Q \supseteq I$ and

$$Q \supseteq \bigcap_{\substack{P \in \text{Spec}^\sharp R \\ P \supseteq I}} P = \sqrt[\sharp]{I} \supseteq \sqrt[\sharp]{J} = \bigcap_{\substack{P \in \text{Spec}^\sharp R \\ P \supseteq J}} P \supseteq J,$$

which proves that $Q \in V^\sharp(J)$. Thus, we get $V^\sharp(I) \subseteq V^\sharp(J)$.

(ii) Since $\sqrt[\sharp]{\sqrt[\sharp]{I}} = \sqrt[\sharp]{I}$, (ii) follows from (i). □

Definition 4.1 Let X be a topological space.

- (i) A closed subset F of X is **reducible** if $F = F_{(1)} \cup F_{(2)}$ for proper closed subsets $F_{(1)}, F_{(2)}$ of X . We call a closed subset F **irreducible** if it is not reducible.
- (ii) X is **quasicompact** if given an arbitrary open covering $\{U_{(i)} \mid i \in I\}$ of X , there exists a finite subcovering of X , i.e., there exist finitely many members $U_{(i_1)}, \dots, U_{(i_n)}$ of $\{U_{(i)} \mid i \in I\}$ such that $X = U_{(i_1)} \cup \dots \cup U_{(i_n)}$.

Using the topological concepts above, we have the following

Proposition 4.3 Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring.

- (i) The trispectrum $\text{Spec}^\sharp R$ is quasicompact.
- (ii) Both $\text{Spec}_0^\sharp R$ and $\text{Spec}_1^\sharp R$ are quasicompact subsets of $\text{Spec}^\sharp R$.
- (iii) If I is a triideal of R , then the closed subset $V^\sharp(I)$ of $\text{Spec}^\sharp R$ is irreducible if and only if $\sqrt[n]{I}$ is a prime triideal.

Proof (i) Let $\{D^\sharp(I_{(i)}) \mid i \in \Delta\}$ be an open covering of $\text{Spec}^\sharp R$, where $I_{(i)} = I_{(i)0} \oplus I_{(i)1}$ is a triideal with the even part $I_{(i)0}$ and the odd part $I_{(i)1}$ for each $i \in \Delta$. Thus, $\text{Spec}^\sharp R = \bigcup_{i \in \Delta} D^\sharp(I_{(i)}) = D^\sharp(\sum_{i \in \Delta} I_{(i)})$. Hence, $V^\sharp(\sum_{i \in \Delta} I_{(i)}) = \emptyset$. If $1 \notin (\sum_{i \in \Delta} I_{(i)})_0 = \sum_{i \in \Delta} I_{(i)0}$, then $(\sum_{i \in \Delta} I_{(i)})_0$ is a proper ideal of the commutative ring $(R_0, +, \cdot)$. Hence, there exists a maximal ideal M_0 of the ring $(R_0, +, \cdot)$ such that $(\sum_{i \in \Delta} I_{(i)})_0 \subseteq M_0$. Since $M_0 \oplus R_1$ is a prime triideal of R and $\sum_{i \in \Delta} I_{(i)} \subseteq M_0 \oplus R_1$, we get that $M_0 \oplus R_1 \in V^\sharp(\sum_{i \in \Delta} I_{(i)}) = \emptyset$, which is impossible. Therefore, $1 \in (\sum_{i \in \Delta} I_{(i)})_0 = \sum_{i \in \Delta} I_{(i)0}$, which implies that $x_{(i_1)0} + x_{(i_2)0} + \cdots + x_{(i_n)0} = 1$ for some positive integer n and $x_{(i_k)0} \in I_{(i_k)0} \subseteq I_{(i_k)}$ with $i_k \in \Delta$ and $n \geq k \geq 1$. It follows that $D^\sharp(x_{(i_k)0}) \subseteq D^\sharp(I_{(i_k)})$ and

$$\begin{aligned} \text{Spec}^\sharp R &= \bigcup_{i \in \Delta} D^\sharp(I_{(i)}) \supseteq \bigcup_{k=1}^n D^\sharp(I_{(i_k)}) \supseteq \bigcup_{k=1}^n D^\sharp(x_{(i_k)0}) \\ &= \bigcup_{k=1}^n D^\sharp(Rx_{(i_k)0}) = D^\sharp\left(\sum_{k=1}^n Rx_{(i_k)0}\right) = D^\sharp(R) = \text{Spec}^\sharp R, \end{aligned}$$

which implies that $\text{Spec}^\sharp R = \bigcup_{k=1}^n D^\sharp(I_{(i_k)})$.

(ii) Note that a closed subset of a quasicompact topological space is a quasicompact subset. Since $\text{Spec}_0^\sharp R = V^\sharp(R_1)$ is a closed subset of the quasicompact topological space $\text{Spec}^\sharp R$, $\text{Spec}_0^\sharp R$ is a quasicompact subset.

It is well-known that a subset C of a topological space X is a quasicompact subset of X if and only if every covering of C by open subsets of X has a finite subcovering. Hence, in order to prove that $\text{Spec}_1^\sharp R$ is a quasicompact subsets of $\text{Spec}^\sharp R$, it suffices to prove that if $\text{Spec}_1^\sharp R = \bigcup_{j \in \Gamma} D^\sharp(J_{(j)})$ for triideals $J_{(j)}$ of R , then there exists a positive integer m such that $\text{Spec}_1^\sharp R = \bigcup_{k=1}^m D^\sharp(J_{(j_k)})$ for some $j_1, \dots, j_k \in \Gamma$.

Since $\text{Spec}_1^\sharp R = \bigcup_{j \in \Gamma} D^\sharp(J_{(j)}) = D^\sharp(\sum_{j \in \Gamma} J_{(j)})$, we have $V^\sharp(\sum_{j \in \Gamma} J_{(j)}) = \text{Spec}^\sharp R \setminus D^\sharp(\sum_{j \in \Gamma} J_{(j)}) = \text{Spec}_0^\sharp R$. If $(\sum_{j \in \Gamma} J_{(j)})_1 \neq R_1$, then there exists a maximal ideal N_1 of the commutative ring $(R_1, +, \cdot)$ such that $(\sum_{j \in \Gamma} J_{(j)})_1 \subseteq N_1$. By Proposition 2.1, there exists an ideal N_0 of the commutative ring $(R_0, +, \cdot)$ such that $N_0 \supseteq (\sum_{j \in \Gamma} J_{(j)})_0$ and $N_0 \oplus N_1$ is a prime triideal of R . Thus, $N_0 \oplus N_1 \in V^\sharp(\sum_{j \in \Gamma} J_{(j)}) = \text{Spec}_0^\sharp R$, which is impossible because $N_1 \neq R_1$. This proves that $(\sum_{j \in \Gamma} J_{(j)})_1 = R_1$. Hence, we have $y_{(j_1)1} + y_{(j_2)1} + \cdots + y_{(j_m)1} = 1$.

$\cdots + y_{(j_m)1} = 1^\sharp$ for some positive integer m and $y_{(j_k)1} \in J_{(j_k)1}$ with $j_k \in \Gamma$ and $m \geq k \geq 1$. It follows that $D^\sharp(y_{(j_k)1}) \subseteq D^\sharp(J_{(j_k)})$ and

$$\begin{aligned} \text{Spec}_1^\sharp R &= \bigcup_{j \in \Gamma} D^\sharp(J_{(j)}) \supseteq \bigcup_{k=1}^m D^\sharp(y_{(j_k)1}) = \bigcup_{k=1}^m D^\sharp(R_1 \sharp y_{(j_k)1}) \\ &= D^\sharp \left(\sum_{k=1}^m R_1 \sharp y_{(j_k)1} \right) = D^\sharp(R_1) = \text{Spec}_1^\sharp R, \end{aligned}$$

which implies that $\text{Spec}_1^\sharp R = \bigcup_{k=1}^m D^\sharp(y_{(j_k)1})$.

(iii) By Proposition 4.2, we may assume $I = \sqrt[m]{I}$ in the following proof. First, we prove that if $V^\sharp(I)$ is irreducible, then I is a prime triideal.

Suppose that $a_0 b_0 \in I$ for some $a_0, b_0 \in R_0$. Let

$$J_{(1)} = I + Ra_0 = (I_0 + R_0 a_0) \oplus (I_1 + R_1 a_0)$$

and

$$K_{(1)} = I + Rb_0 = (I_0 + R_0 b_0) \oplus (I_1 + R_1 b_0).$$

Then both $J_{(1)}$ and $K_{(1)}$ are triideals of R and

$$\begin{aligned} J_{(1)} \sharp K_{(1)} &= (I_0 + R_0 a_0)(I_0 + R_0 b_0) + (I_1 + R_1 a_0) \sharp (I_1 + R_1 b_0) \\ &\subseteq I + R_0 a_0 R_0 b_0 + (R_1 a_0) \sharp (R_1 b_0) \subseteq I + R_0 a_0 b_0 + R_1 a_0 b_0 \subseteq I. \end{aligned}$$

Hence, we get $V^\sharp(J_{(1)}) \cup V^\sharp(K_{(1)}) = V^\sharp(J_{(1)} \sharp K_{(1)}) \supseteq V^\sharp(I)$ by Proposition 4.1 (ii). It is clear that $V^\sharp(J_{(1)}) \subseteq V^\sharp(I)$ and $V^\sharp(K_{(1)}) \subseteq V^\sharp(I)$. Hence, we get that $V^\sharp(J_{(1)}) \cup V^\sharp(K_{(1)}) \subseteq V^\sharp(I)$. Thus we have $V^\sharp(J_{(1)}) \cup V^\sharp(K_{(1)}) = V^\sharp(I)$. Since $V^\sharp(I)$ is irreducible, $V^\sharp(J_{(1)}) = V^\sharp(I)$ or $V^\sharp(K_{(1)}) = V^\sharp(I)$, which imply that $I = \sqrt[m]{J_{(1)}} \supseteq J_{(1)} \ni a_0$ or $I = \sqrt[m]{K_{(1)}} \supseteq K_{(1)} \ni b_0$. This proves that

$$a_0 b_0 \in I \implies a_0 \in I \text{ or } b_0 \in I \text{ for } a_0, b_0 \in R_0. \quad (40)$$

Suppose that $a_0 b_1 \in I$ for some $a_0 \in R_0$ and $b_1 \in R_1$. Using the triideals

$$J_{(2)} = I + Ra_0 = (I_0 + R_0 a_0) \oplus (I_1 + R_1 a_0)$$

and

$$K_{(2)} = I + R_1 \sharp b_1 = I_0 \oplus (I_1 + R_1 \sharp b_1),$$

we have

$$\begin{aligned} J_{(2)} \sharp K_{(2)} &= (I_0 + R_0 a_0)I_0 + (I_1 + R_1 a_0) \sharp (I_1 + R_1 \sharp b_1) \\ &\subseteq I + (R_1 a_0) \sharp R_1 \sharp b_1 \subseteq I + (a_0 R_1) \sharp b_1 \sharp R_1 \subseteq I + a_0 (R_1 \sharp b_1) \sharp R_1 \\ &\subseteq I + a_0 (b_1 \sharp R_1) \sharp R_1 \subseteq I + (a_0 b_1) \sharp R_1 \sharp R_1 \subseteq I, \end{aligned}$$

which implies that $I = \sqrt[3]{J_{(2)}} \supseteq J_{(2)} \ni a_0$ or $I = \sqrt[3]{K_{(2)}} \supseteq K_{(2)} \ni b_1$. This proves that

$$a_0 b_1 \in I \implies a_0 \in I \text{ or } b_1 \in I \text{ for } a_0 \in R_0 \text{ and } b_1 \in R_1. \quad (41)$$

Similarly, we have

$$a_1 b_0 \in I \implies a_1 \in I \text{ or } b_0 \in I \text{ for } a_1 \in R_1 \text{ and } b_0 \in R_0 \quad (42)$$

and

$$a_1 \# b_1 \in I \implies a_1 \in I \text{ or } b_1 \in I \text{ for } a_1, b_1 \in R_1. \quad (43)$$

By (40), (41), (42) and (43), I is a prime triideal.

Next, we prove that if I is a prime triideal, then $V^\#(I)$ is irreducible. Suppose that $V^\#(I) = V^\#(J) \cup V^\#(K)$, where J and K are triideals of R . Using Proposition 4.2 (ii), we can assume that $\sqrt[3]{J} = J$ and $\sqrt[3]{K} = K$. In this case, we have

$$V^\#(J) \subseteq V^\#(I) \implies I = \sqrt[3]{I} \subseteq \sqrt[3]{J} = J$$

and

$$V^\#(K) \subseteq V^\#(I) \implies I = \sqrt[3]{I} \subseteq \sqrt[3]{K} = K.$$

By Proposition 4.1 (ii), we have $V^\#(I) = V^\#(J) \cup V^\#(K) = V^\#(J \dot{\#} K)$. This fact and Proposition 4.2 (i) give $J \dot{\#} K \subseteq \sqrt[3]{I} = I$. Since I is a prime triideal, we get $J \subseteq I$ or $K \subseteq I$ by Proposition 2.2. Hence, $I = J$ or $I = K$. Thus, $V^\#(I) = V^\#(J)$ or $V^\#(I) = V^\#(K)$. This proves that $V^\#(I)$ is irreducible. \square

5 Localization of Hu-Liu Trirings

Let $(R = R_0 \oplus R_1, +, \cdot, \#)$ be a Hu-Liu triring with the identity 1 and the local identity $1^\#$. A subset S of R is called a **multiplicative subset** if

$$0 \notin S, \quad 1 \in S_0 := S \cap R_0, \quad 1^\# \in S_1 := S \cap R_1, \quad S = S_0 \cup S_1$$

and

$$s_0 t_0 \in S_0, \quad s_0 s_1 \in S_1, \quad s_1 s_0 \in S_1, \quad s_1 \# t_1 \in S_1, \quad s_0 S_1 = S_1 s_0, \quad (44)$$

where $s_i, t_i \in S_i$ and $i = 0, 1$.

Given a multiplicative subset S of a Hu-Liu triring R , we define a relation in the Cartesian product $(R_0 \times S_0) \times (R_1 \times S_1)$ as follows:

$$\begin{aligned} & ((a_0, s_0), (a_1, s_1)) \sim ((b_0, t_0), (b_1, t_1)) \\ \Leftrightarrow & u_0(a_0 t_0 - b_0 s_0) = 0 \quad \text{and} \quad u_1 \# (a_1 \# t_1 - b_1 \# s_1) = 0 \end{aligned} \quad (45)$$

for some $u_i \in S_i$, where $a_i, b_i \in R_i, s_i, t_i \in S_i$ and $i = 0, 1$.

It is clear that \sim is an equivalence relation. Let $\left(\frac{a_0}{s_0}, \frac{a_1}{s_1}\right)$ be the equivalence class containing $((a_0, s_0), (a_1, s_1))$; that is,

$$\left(\frac{a_0}{s_0}, \frac{a_1}{s_1}\right) := \left\{ ((a'_0, s'_0), (a'_1, s'_1)) \left| \begin{array}{l} (a'_i, s'_i) \in R_i \times S_i, \\ u_0(a_0 s'_0 - s_0 a'_0) = 0, \\ u_1 \# (a_1 \# s'_1 - s_1 \# a'_1) = 0 \\ \text{for some } u_i \in S_i \text{ with } i = 0, 1. \end{array} \right. \right\}.$$

Let

$$S^{-1}R := \left\{ \left(\frac{a_0}{s_0}, \frac{a_1}{s_1}\right) \left| ((a_0, s_0), (a_1, s_1)) \in (R_0 \times S_0) \times (R_1 \times S_1) \right. \right\}$$

be the set of all equivalence classes. Then $S^{-1}R = (S^{-1}R)_0 \oplus (S^{-1}R)_1$ is a Hu-Liu triring, where the even part $(S^{-1}R)_0$ and odd part $(S^{-1}R)_1$ are given by

$$\begin{aligned} (S^{-1}R)_0 &:= \left\{ \left(\frac{a_0}{s_0}, \frac{0}{1^\#}\right) \left| (a_0, s_0) \in R_0 \times S_0 \right. \right\}, \\ (S^{-1}R)_1 &:= \left\{ \left(\frac{0}{1}, \frac{a_1}{s_1}\right) \left| (a_1, s_1) \in R_1 \times S_1 \right. \right\} \end{aligned}$$

and the addition $+$, the ring multiplication \cdot and the local product $\#$ are given by

$$\begin{aligned} \left(\frac{a_0}{s_0}, \frac{a_1}{s_1}\right) + \left(\frac{b_0}{t_0}, \frac{b_1}{t_1}\right) &:= \left(\frac{a_0 t_0 + s_0 b_0}{s_0 t_0}, \frac{a_1 \# t_1 + s_1 \# b_1}{s_1 \# t_1}\right), \\ \left(\frac{a_0}{s_0}, \frac{a_1}{s_1}\right) \cdot \left(\frac{b_0}{t_0}, \frac{b_1}{t_1}\right) &:= \left(\frac{a_0 b_0}{s_0 t_0}, \frac{(a_0 b_1) \# (s_1 t_0) + (s_0 t_1) \# (a_1 b_0)}{(s_0 t_1) \# (s_1 t_0)}\right) \end{aligned}$$

and

$$\left(\frac{0}{1}, \frac{a_1}{s_1}\right) \# \left(\frac{0}{1}, \frac{b_1}{t_1}\right) := \left(\frac{0}{1}, \frac{a_1 \# b_1}{s_1 \# t_1}\right).$$

Clearly, $\left(\frac{1}{1}, \frac{0}{1^\#}\right)$ is the identity of the ring $S^{-1}R$, and $\left(\frac{0}{1}, \frac{1^\#}{1^\#}\right)$ is the local identity.

Let $i_R^{\#S} : R \rightarrow S^{-1}R$ be the map defined by

$$i_R^{\#S} : a = a_0 + a_1 \mapsto \left(\frac{a_0}{1}, \frac{a_1}{1^\#}\right),$$

where $a_i \in R_i$ for $i = 0$ and 1 . Then $i_R^{\#S}$ is a triring homomorphism from R to $S^{-1}R$.

The pair $(S^{-1}R, i_R^{\sharp S})$ constructed above is called the **localization** of R with respect to S , and $i_R^{\sharp S}$ is called the **canonical triring homomorphism** from R to $S^{-1}R$. The next proposition gives the universal mapping characterization of the localization $(S^{-1}R, i_R^{\sharp S})$.

Proposition 5.1 *Let $R = R_0 \oplus R_1$ be a Hu-Liu triring. If $S = S_0 \cup S_1$ is a multiplicative subset of R , then the localization $(S^{-1}R, i_R^{\sharp S})$ of R with respect to S has the following two properties.*

- (i) *For any $s_i \in S_i$ with $i = 0$ and 1 , $i_R^{\sharp S}(s_0)$ is invertible in the commutative ring $\left((S^{-1}R)_0, +, \cdot\right)$ and $i_R^{\sharp S}(s_1)$ is invertible in the commutative ring $\left((S^{-1}R)_1, +, \sharp\right)$.*
- (ii) *If $\psi : R \rightarrow \overline{R}$ is a triring homomorphism from the Hu-Liu triring R to a Hu-Liu triring \overline{R} such that $\psi(s_0)$ is invertible in the commutative ring $(\overline{R}_0, +, \cdot)$ and $\psi(s_1)$ is invertible in the commutative ring $(\overline{R}_1, +, \sharp)$ for any $s_i \in S_i$ with $i = 0$ and 1 , then there exists a unique triring homomorphism $\overline{\psi} : S^{-1}R \rightarrow \overline{R}$ such that $\psi = \overline{\psi} i_R^{\sharp S}$.*

Proof A direct computation. □

Let $R = R_0 \oplus R_1$ be a Hu-Liu triring. The following three types of localizations of R are very useful in the study of Hu-Liu trirings:

Type 1. If $P = P_0 \oplus P_1$ is a triideal of R with $P_1 \neq R_1$, then $(R_0 \setminus P_0) \cup (R_1 \setminus P_1)$ is a multiplicative subset of R . The localization of R with respect to $(R_0 \setminus P_0) \cup (R_1 \setminus P_1)$ is called the **localization** of R at P and is denoted by R_P^{\sharp} .

Type 2. If $f_0 \in R_0$ and $D^{\sharp}(f_0) \cap \text{spec}_1^{\sharp} R \neq \emptyset$, then $T(f_0) = T_0(f_0) \cup T_1(f_0)$ is a multiplicative subset of R , where $T_0(f_0) := \{f_0^n \mid n \in \mathbb{Z}_{\geq 0}\}$ and

$$T_1(f_0) := \{z_1 f_0^n \mid n \in \mathbb{Z}_{\geq 0} \text{ and } z_1 \in R_1 \setminus P \text{ for all } P \in D^{\sharp}(f_0) \cap \text{spec}_1^{\sharp} R\}.$$

The localization of R with respect to $T(f_0) = T_0(f_0) \cup T_1(f_0)$ is called the **localization of R at f_0** and is denoted by $R_{f_0}^{\sharp}$. Clearly, we have

$$R_{f_0}^{\sharp} = \left\{ \frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m} \mid m, n \geq 0, a_i \in R_i \text{ for } i = 0, 1 \text{ and } z_1 f_0^m \in T_1(f_0) \right\}.$$

Type 3. If $f_1 \in R_1$ and f_1 is not trinilpotent, then $\{1\} \cup \{f_1^{\sharp m}\}_{m \geq 0}$ is multiplication subset of R . The localization of R with respect to $\{1\} \cup \{f_1^{\sharp m}\}_{m \geq 0}$ is called the **localization of R at f_1** and is denoted by $R_{f_1}^{\sharp}$. Thus, we have

$$R_{f_1}^{\sharp} = \left\{ a_0 + \frac{a_1}{f_1^{\sharp m}} \mid m, n \geq 0 \text{ and } a_i \in R_i \text{ for } i = 0, 1 \right\}.$$

6 Sheaf Structures on Trispectra

We begin this section with the following

Definition 6.1 Let X be a topological space. A **presheaf** \mathcal{P} of Hu-Liu trirings on X assigns a Hu-Liu triring $\mathcal{P}(U)$ to each open set U of X and a triring homomorphism: $\rho_{UV} : \mathcal{P}(U) \rightarrow \mathcal{P}(V)$, called the **restriction map**, to each inclusion of open sets $V \subseteq U$, subject to the conditions:

- (i) $\mathcal{P}(\emptyset) = \emptyset$;
- (ii) $\rho_{UU} : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$ is the identity map;
- (iii) If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \rho_{UV}$.

If \mathcal{P} is a presheaf of Hu-Liu trirings on a topological space X , then an element $s \in \mathcal{P}(U)$ is called a **section** of \mathcal{P} over the open set U .

Definition 6.2 A presheaf \mathcal{P} of Hu-Liu trirings on a topological space X is called a **sheaf** of Hu-Liu trirings if it satisfies the following two conditions for an arbitrary open subset U of X and an arbitrary open covering $\{U_{(i)} \mid i \in \Delta\}$ of U :

- (i) If $s \in \mathcal{P}(U)$ and $\rho_{U_{(i)}}(s) = 0$ for all $i \in \Delta$, then $s = 0$;
- (ii) If we have elements $s_{(i)} \in \mathcal{P}(U_{(i)})$ for each $i \in \Delta$ having the property that $\rho_{U_{(i)}, U_{(i)} \cap U_{(j)}}(s_{(i)}) = \rho_{U_{(j)}, U_{(i)} \cap U_{(j)}}(s_{(j)})$ for $i, j \in \Delta$, then there is an element $s \in \mathcal{P}(U)$ such that $\rho_{U_{(i)}}(s) = s_{(i)}$ for each $i \in \Delta$.

Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring with the identity 1 and the local identity 1^{\sharp} . If $f_0 \in R_0$ and $f_1 \in R_1$, then we define

$$\mathcal{O}(D^{\sharp}(f_0)) := \begin{cases} (R_0)_{f_0} \oplus R_0 & \text{if } f_0 \notin \sqrt[3]{0} \text{ and } D^{\sharp}(f_0) \cap \text{spec}_1^{\sharp} R = \emptyset, \\ R_{f_0}^{\sharp} \oplus R_{1^{\sharp} f_0}^{\sharp} & \text{if } f_0 \notin \sqrt[3]{0} \text{ and } D^{\sharp}(f_0) \cap \text{spec}_1^{\sharp} R \neq \emptyset, \\ 0 & \text{if } f_0 \in \sqrt[3]{0} \end{cases}$$

and

$$\mathcal{O}(D^\sharp(f_1)) := \begin{cases} R_{f_1}^\sharp & \text{if } f_1 \notin \sqrt[n]{0}, \\ 0 & \text{if } f_1 \in \sqrt[n]{0} \end{cases},$$

where $(R_0)_{f_0}$ is the ordinary localization of the commutative ring $(R_0, +, \cdot)$ with respect to the multiplication set $\{f_0^n \mid n \in \mathbb{Z}_{\geq 0}\}$, and $(R_0)_{f_0} \oplus R_0$ is a Hu-Liu triring with zero odd part. We then have the following result.

Proposition 6.1 *Let $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ be a Hu-Liu triring. If $f_i, g_i \in R_i$ for $i = 0, 1$ and $D^\sharp(g_i) \subseteq D^\sharp(f_j)$ for $i, j \in \{0, 1\}$, then there exist a triring homomorphism $\rho_{f_j, g_i} : \mathcal{O}(D^\sharp(f_j)) \rightarrow \mathcal{O}(D^\sharp(g_i))$ such that*

$$(i) \quad \rho_{f_j, f_j} = id_{D^\sharp(f_j)},$$

$$(ii) \quad \rho_{f_j, h_k} = \rho_{g_i, h_k} \rho_{f_j, g_i} \text{ provided } D^\sharp(h_k) \subseteq D^\sharp(g_i) \subseteq D^\sharp(f_j),$$

where $i, j, k \in \{0, 1\}$.

Proof If $g_i \in \sqrt[n]{0}$, then we define $\rho_{f_j, g_i} : \mathcal{O}(D^\sharp(f_j)) \rightarrow \mathcal{O}(D^\sharp(g_i))$ to be the zero homomorphism, i.e.,

$$\rho_{f_j, g_i}(x) := 0 \quad \text{for } x \in \mathcal{O}(D^\sharp(f_j)) \text{ and } g_i \in \sqrt[n]{0}. \quad (46)$$

We now define $\rho_{f_j, g_i} : \mathcal{O}(D^\sharp(f_j)) \rightarrow \mathcal{O}(D^\sharp(g_i))$ for $g_i \notin \sqrt[n]{0}$ and $f_j \notin \sqrt[n]{0}$ by cases.

Case 1: $i = 0, j = 0$, in which case, we have

$$\begin{aligned} D^\sharp(Rg_0) &= D^\sharp(g_0) \subseteq D^\sharp(f_0) = D^\sharp(Rf_0) \\ \implies V^\sharp(Rg_0) &\supseteq V^\sharp(Rf_0) \implies \sqrt[n]{Rg_0} \subseteq \sqrt[n]{Rf_0} \quad (\text{by Proposition 4.2 (i)}) \\ \implies g_0^u &= r_0 f_0 \quad \text{for some } u \in \mathbb{Z}_{>0} \text{ and } r_0 \in R_0. \end{aligned} \quad (47)$$

Since $D^\sharp(g_0) \cap \text{spec}_1^\sharp R \subseteq D^\sharp(f_0) \cap \text{spec}_1^\sharp R$, we have three subcases.

Case 1(i): $D^\sharp(g_0) \cap \text{spec}_1^\sharp R = \emptyset$ and $D^\sharp(f_0) \cap \text{spec}_1^\sharp R = \emptyset$. In this case, using (47) and universal property of the ordinary localization $(R_0)_{f_0}$, we get a triring homomorphism

$$\rho_{f_0, g_0} : \mathcal{O}(D^\sharp(f_0)) = (R_0)_{f_0} \oplus R_0 \rightarrow (R_0)_{g_0} \oplus R_0 = \mathcal{O}(D^\sharp(g_0))$$

such that

$$\rho_{f_0, g_0} \left(\frac{a_0}{f_0^n}, b_0 \right) = \left(\frac{a_0 r_0^n}{g_0^{nu}}, b_0 \right) \quad \text{for } a_0, b_0 \in R_0. \quad (48)$$

Case 1(ii): $D^\sharp(g_0) \cap \text{spec}_1^\sharp R = \emptyset$ and $D^\sharp(f_0) \cap \text{spec}_1^\sharp R \neq \emptyset$. In this case, we have a triring homomorphism

$$\rho_{f_0, g_0} : \mathcal{O}(D^\sharp(f_0)) = R_{f_0}^\sharp \oplus R_{1^\sharp f_0}^\sharp \rightarrow (R_0)_{g_0} \oplus R_0 = \mathcal{O}(D^\sharp(g_0))$$

such that

$$\rho_{f_0, g_0} \left(\left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\# f_0^k} \right) \right) = \left(\frac{a_0 r_0^n}{g_0^{nu}}, b_0 \right), \quad (49)$$

where $z_1 f_0^m \in T_1(f_0)$, $n, m, k \in \mathbb{Z}_{\geq 0}$, $a_i, b_i \in R_i$ and $i = 0, 1$.

Case 1(iii): $D^\#(g_0) \cap \text{spec}_1^\# R \neq \emptyset$ and $D^\#(f_0) \cap \text{spec}_1^\# R \neq \emptyset$. Using (47) and universal property of the localization $R_{f_0}^\#$ and $R_{1^\# f_0}^\#$, we get two triring homomorphism $\bar{\phi} : R_{f_0}^\# \rightarrow R_{g_0}^\#$ and $\bar{\psi} : R_{1^\# f_0}^\# \rightarrow R_{1^\# g_0}^\#$ such that

$$\bar{\phi} \left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m} \right) = \frac{a_0 r_0^n}{g_0^{nu}} + \frac{a_1 r_0^m}{z_1 g_0^{mu}} \quad (50)$$

and

$$\bar{\psi} \left(b_0 + \frac{b_1}{1^\# f_0^k} \right) = b_0 + \frac{b_1 r_0^k}{1^\# g_0^{ku}}, \quad (51)$$

where $z_1 f_0^m \in T_1(f_0)$, $n, m, k \in \mathbb{Z}_{\geq 0}$, $a_i, b_i \in R_i$ and $i = 0, 1$.

By (50) and (51), we get a triring homomorphism

$$\rho_{f_0, g_0} : \mathcal{O}(D^\#(f_0)) = R_{f_0}^\# \oplus R_{1^\# f_0}^\# \rightarrow R_{g_0}^\# \oplus R_{1^\# g_0}^\# = \mathcal{O}(D^\#(g_0))$$

such that

$$\begin{aligned} & \rho_{f_0, g_0} \left(\left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\# f_0^k} \right) \right) \\ &= \left(\bar{\phi} \left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m} \right), \bar{\psi} \left(b_0 + \frac{b_1}{1^\# f_0^k} \right) \right) \\ &= \left(\frac{a_0 r_0^n}{g_0^{nu}} + \frac{a_1 r_0^m}{z_1 g_0^{mu}}, b_0 + \frac{b_1 r_0^k}{1^\# g_0^{ku}} \right), \end{aligned} \quad (52)$$

where $z_1 f_0^m \in T_1(f_0)$, $n, m, k \in \mathbb{Z}_{\geq 0}$, $a_i, b_i \in R_i$ and $i = 0, 1$.

Case 2: $i = 0$ and $j = 1$, in which case, we have

$$D^\#(g_0) \subseteq D^\#(f_1) \implies g_0 \in \sqrt[3]{0}. \quad (53)$$

Hence, $\rho_{f_1, g_0} = 0$ in this case according to the definition given at the beginning.

Case 3: $i = 1$ and $j = 0$, in which case,

$$\begin{aligned} & D^\#(g_1) \subseteq D^\#(f_0) \\ \implies & g_1^{\#v} = r_1 f_0 \quad \text{for some } v \in \mathbb{Z}_{>0} \text{ and } r_1 \in R_1 \end{aligned} \quad (54)$$

and $D^\#(f_0) \cap \text{spec}_1^\# R \supseteq D^\#(g_1) \neq \emptyset$. It follows that we have a triring homomorphism

$$\rho_{f_0, g_1} : \mathcal{O}(D^\#(f_0)) = R_{f_0}^\# \oplus R_{1^\# f_0}^\# \rightarrow R_{g_1}^\# = \mathcal{O}(D^\#(g_1))$$

such that

$$\rho_{f_0, g_0} \left(\left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\# f_0^k} \right) \right) = b_0 + \frac{b_1^\# r_1^{\#k}}{g_1^{\#kv}}, \quad (55)$$

where $z_1 f_0^m \in T_1(f_0)$, $n, m, k \in \mathbb{Z}_{\geq 0}$, $a_i, b_i \in R_i$ and $i = 0, 1$.

Case 4: $i = 1$ and $j = 1$, in which case,

$$\begin{aligned} D^\#(g_1) &\subseteq D^\#(f_1) \\ \implies g_1^{\#\theta} &= t_1^\# f_1 \quad \text{for some } \theta \in \mathbb{Z}_{>0} \text{ and } t_1 \in R_1. \end{aligned} \quad (56)$$

It follows that we have a triring homomorphism

$$\rho_{f_1, g_1} : \mathcal{O}(D^\#(f_1)) = R_{f_1}^\# \rightarrow R_{g_1}^\# = \mathcal{O}(D^\#(g_1))$$

such that

$$\rho_{f_1, g_1} \left(b_0 + \frac{b_1}{f_1^{\#n}} \right) = b_0 + \frac{b_1^\# t_1^{\#n}}{g_1^{\#\theta n}}, \quad (57)$$

where $n \in \mathbb{Z}_{\geq 0}$, $b_i \in R_i$ and $i = 0, 1$.

It is clear that the triring homomorphisms ρ_{f_j, g_i} defined by (46), (52), (55) and (57) has the property (i).

Note that if $h_k \in \sqrt[n]{0}$, then the property (ii) holds because $\rho_{f_j, h_k} = \rho_{g_i, h_k} = 0$. If $h_k \notin \sqrt[n]{0}$, then we have $k \geq i \geq j$ by (53). Therefore, there are only four cases:

$$(k, i, j) = (0, 0, 0) \text{ or } (1, 0, 0) \text{ or } (1, 1, 0) \text{ or } (1, 1, 1).$$

One can check that the property (ii) also holds for each of the four cases above. \square

Let U be a nonempty open subset of $\text{Spec}^\# R$, and let

$$S_U := \{ D^\#(f_{(\alpha)}) \mid \emptyset \neq D^\#(f_{(\alpha)}) \subseteq U \text{ and } \alpha \in \Lambda_U \}$$

be the set of the nonempty basic open subsets contained in U . The set Λ_U is a partial order set with respect to the following partial order:

$$\alpha \geq \beta \quad \text{if and only if} \quad D^\#(f_{(\alpha)}) \supseteq D^\#(f_{(\beta)}).$$

$(\mathcal{O}(D^\#(f_{(\alpha)})), \rho_{f_{(\alpha)}, f_{(\beta)}})$ is an **inverse system** on the set Λ_U , where the triring homomorphism

$$\rho_{f_{(\alpha)}, f_{(\beta)}} : \mathcal{O}(D^\#(f_{(\alpha)})) \rightarrow \mathcal{O}(D^\#(f_{(\beta)})) \quad \text{for } \alpha \geq \beta$$

is defined by Proposition 6.1. We define

$$\mathcal{O}(U) := \varprojlim_{\Lambda_U} \mathcal{O}(D^\#(f_{(\alpha)})), \quad (58)$$

where $\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)}))$ is the **inverse limit** of the inverse system $(\mathcal{O}(D^\sharp(f_{(\alpha)})), \rho_{f_{(\alpha)}, f_{(\beta)}})$. The inverse limit is a subtring of the direct product $\prod_{\alpha \in \Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)}))$, which is given by

$$\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})) = \left\{ (x_{(\alpha)})_{\alpha \in \Lambda_U} \left| \begin{array}{l} x_{(\alpha)} \in \mathcal{O}(D^\sharp(f_{(\alpha)})) \\ \text{and } x_{(\beta)} = \rho_{f_{(\alpha)}, f_{(\beta)}}(x_{(\alpha)}) \\ \text{whenever } \alpha \geq \beta \end{array} \right. \right\}. \quad (59)$$

The even part and odd part of $\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)}))$ are given by

$$\left(\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})) \right)_i = \left\{ (x_{(\alpha)i})_{\alpha \in \Lambda_U} \left| \begin{array}{l} x_{(\alpha)i} \in (\mathcal{O}(D^\sharp(f_{(\alpha)})))_i \\ \text{and } x_{(\beta)i} = \rho_{f_{(\alpha)}, f_{(\beta)}}(x_{(\alpha)i}) \\ \text{whenever } \alpha \geq \beta \end{array} \right. \right\}, \quad (60)$$

where $i = 0, 1$. For each $\alpha \in \Lambda_U$, let $p_{(\alpha)}^U : \varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})) \rightarrow \mathcal{O}(D^\sharp(f_{(\alpha)}))$ be the triring homomorphism defined by

$$p_{(\alpha)}^U((x_{(\gamma)})_{\gamma \in \Lambda_U}) := x_{(\alpha)}. \quad (61)$$

Clearly, we have

$$p_{(\beta)}^U = \rho_{f_{(\alpha)}, f_{(\beta)}} p_{(\alpha)}^U \quad \text{if } \alpha \geq \beta \text{ and } \alpha, \beta \in \Lambda_U. \quad (62)$$

The pair $\left(\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})), \{p_{(\alpha)}^U \mid \alpha \in \Lambda_U\} \right)$ has the following universal property:

Let $(X, \{q_{(\alpha)} \mid \alpha \in \Lambda_U\})$ be a pair consisting of a triring X and a family of triring homomorphism

$$\{q_{(\alpha)} : X \rightarrow \mathcal{O}(D^\sharp(f_{(\alpha)})) \mid \alpha \in \Lambda_U\}.$$

If $q_{(\beta)} = \rho_{f_{(\alpha)}, f_{(\beta)}} q_{(\alpha)}$ whenever $\alpha \geq \beta$, then there exists a unique triring homomorphism $\sigma : X \rightarrow \varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)}))$ such that $q_{(\alpha)} = p_{(\alpha)}^U \sigma$ for each $\alpha \in \Lambda_U$.

If $V \subseteq U$, then $S_V \subseteq S_U$ and $\Lambda_V \subseteq \Lambda_U$. Consider the pair

$$\left(\varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})), \{p_{(\alpha)}^U \mid \alpha \in \Lambda_V\} \right).$$

Note that (62) is clearly true for $\alpha, \beta \in \Lambda_V \subseteq \Lambda_U$ with $\alpha \geq \beta$. By the universal property of the following pair

$$\left(\varprojlim_{\Lambda_V} \mathcal{O}(D^\sharp(f_{(\alpha)})), \{p_{(\alpha)}^U \mid \alpha \in \Lambda_V\} \right),$$

we have a unique triring homomorphism

$$\rho_{U,V} : \mathcal{O}(U) = \varprojlim_{\Lambda_U} \mathcal{O}(D^\sharp(f_{(\alpha)})) \rightarrow \varprojlim_{\Lambda_V} \mathcal{O}(D^\sharp(f_{(\alpha)})) = \mathcal{O}(V) \quad (63)$$

such that

$$p_{(\alpha)}^U = p_{(\alpha)}^V \rho_{U,V} \quad \text{for } \alpha \in \Lambda_V. \quad (64)$$

It follows from (63) and (64) that

$$\rho_{U,U} = id_{\mathcal{O}(U)} \quad \text{for each open subset } U \text{ of } Spec^\sharp R \quad (65)$$

and

$$\rho_{U,W} = \rho_{V,W} \rho_{U,V} \quad \text{for three open subsets } W \subseteq V \subseteq U. \quad (66)$$

By (65) and (66), \mathcal{O} is a presheaf with the restriction map $\rho_{U,V}$ given by (63). This presheaf is called the **structure presheaf** on $Spec^\sharp R$.

Proposition 6.2 *If $(R = R_0 \oplus R_1, +, \cdot, \sharp)$ is a Hu-Liu triring, then the structure presheaf on $Spec^\sharp R$ is a sheaf of Hu-Liu trirings.*

Proof According to the definition of the structure presheaf on $Spec^\sharp R$, it suffices to prove that the properties (i) and (ii) in Definition 6.2 holds for an arbitrary basic open set and an arbitrary open covering which consists of basic open sets. Let $D^\sharp(f_i)$ be an arbitrary basic open set of $Spec^\sharp R$. Consider the following arbitrary open covering of $D^\sharp(f_i)$

$$D^\sharp(f_i) = \left(\bigcup_{\alpha \in \Lambda_0} D^\sharp(g_{(\alpha)0}) \right) \bigcup \left(\bigcup_{\beta \in \Lambda_1} D^\sharp(g_{(\beta)1}) \right), \quad (67)$$

where $f_0, g_{(\alpha)0} \in R_0$ for $\alpha \in \Lambda_0$, $f_1, g_{(\beta)1} \in R_1$ for $\beta \in \Lambda_1$, Λ_0 and Λ_1 are two index sets.

To prove the properties (i) in Definition 6.2, we need to prove that if $s \in \mathcal{O}(D^\sharp(f_i))$ satisfies

$$\rho_{f_i, g_{(\alpha)0}}(s) = 0 \quad \text{for } \alpha \in \Lambda_0 \quad (68)$$

and

$$\rho_{f_i, g_{(\beta)1}}(s) = 0 \quad \text{for } \beta \in \Lambda_1, \quad (69)$$

then $s = 0$.

Case 1: $i = 0$, in which case, $\Lambda_0 \neq \emptyset$. Since $s \in \mathcal{O}(D^\sharp(f_0))$, we have

$$s = \left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\sharp f_0^k} \right) \quad (70)$$

for some $n, m, k \in \mathbb{Z}_{\geq 0}$, $z_1 f_0^m \in T_1(f_0)$, $a_i, b_i \in R_i$ and $i = 0, 1$. Since $D^\sharp(g_{(\alpha)0}) \subseteq D^\sharp(f_0)$, there exist some $r_{(\alpha)0} \in R_0$ and $u_{(\alpha)} \in \mathbb{Z}_{>0}$ such that

$$g_{(\alpha)0}^{u_{(\alpha)}} = r_{(\alpha)0} f_0 \quad \text{for each } \alpha \in \Lambda_0. \quad (71)$$

- If $D^\sharp(f_0) \cap \text{Spec}_1^\sharp R \neq \emptyset$, then for each $\alpha \in \Lambda_0$, we have

$$\text{either } D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R \neq \emptyset \quad \text{or} \quad D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R = \emptyset.$$

Using (52), (49), (68) and (71), we have

$$\begin{aligned} 0 &= \rho_{f_0, g_{(\alpha)0}}(s) = \rho_{f_0, g_{(\alpha)0}} \left(\left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\sharp f_0^k} \right) \right) \\ &= \begin{cases} \left(\frac{a_0 r_{(\alpha)0}^n}{g_0^{nu(\alpha)}} + \frac{a_1 r_{(\alpha)0}^m}{z_1 g_0^{mu(\alpha)}}, b_0 + \frac{b_1 r_{(\alpha)0}^k}{1^\sharp g_0^{ku(\alpha)}} \right) & \text{if } D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R \neq \emptyset \\ \left(\frac{a_0 r_{(\alpha)0}^n}{g_0^{nu(\alpha)}}, b_0 \right) & \text{if } D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R = \emptyset \end{cases} \end{aligned} \quad (72)$$

- If $D^\sharp(f_0) \cap \text{Spec}_1^\sharp R = \emptyset$, then we have $D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R = \emptyset$ for each $\alpha \in \Lambda_0$. By (49), (68) and (71), we have

$$\begin{aligned} 0 &= \rho_{f_0, g_{(\alpha)0}}(s) = \rho_{f_0, g_{(\alpha)0}} \left(\left(\frac{a_0}{f_0^n} + \frac{a_1}{z_1 f_0^m}, b_0 + \frac{b_1}{1^\sharp f_0^k} \right) \right) \\ &= \left(\frac{a_0 r_{(\alpha)0}^n}{g_0^{nu(\alpha)}}, b_0 \right) \quad \text{for each } \alpha \in \Lambda_0 \end{aligned} \quad (73)$$

Note that

$$D^\sharp(g_{(\alpha)0}) \cap \text{Spec}_1^\sharp R = \emptyset \implies 1^\sharp g_{(\alpha)0}^b = 0 \quad \text{for some } b \in \mathcal{Z}_{\geq 0} \quad (74)$$

It follows from (72), (73), (74) and Proposition 4.2 (i) that there exist $t, d, \delta_{\alpha_i} \in \mathcal{Z}_{>0}$, $\alpha_i \in \Lambda_0$ and $h_{(\alpha_i)0} \in R_0$ such that

$$f_0^t = \sum_{i=1}^d h_{(\alpha_i)0} g_{(\alpha_i)0}^{\delta_{\alpha_i}} \quad (75)$$

and

$$f_0^t a_0 = (1^\sharp f_0^t)^\sharp a_1 = b_0 = (1^\sharp f_0^t)^\sharp b_1 = 0. \quad (76)$$

By (76), $\frac{a_0}{f_0^n} + \frac{a_1}{1^\sharp f_0^m} = 0$ and $b_0 + \frac{b_1}{1^\sharp f_0^k} = 0$. This proves $s = 0$.

Case 2: $i = 1$, in which case, $s \in \mathcal{O}(D^\sharp(f_1))$. Thus $s = a_0 + \frac{a_1}{f_1^{\sharp m}}$ for some $a_i \in R_i$ and $m \in \mathcal{Z}_{\geq 0}$. By (53), $D^\sharp(g_{(\alpha)0}) = \emptyset$ for $\alpha \in \Lambda_0$ in this case. Thus, (67) becomes

$$D^\sharp(f_1) = \bigcup_{\beta \in \Lambda_1} D^\sharp(g_{(\beta)1}). \quad (77)$$

Since $D^\sharp(g_{(\beta)1}) \subseteq D^\sharp(f_1)$, there exist some $r_{(\beta)1} \in R_1$ and $\theta_{(\beta)} \in \mathcal{Z}_{>0}$ such that

$$g_{(\beta)1}^{\theta_{(\beta)}} = r_{(\beta)1}^\sharp f_1 \quad \text{for each } \beta \in \Lambda_1. \quad (78)$$

Using (57) and (78), we have

$$0 = \rho_{f_1, g_{(\beta)_1}}(s) = \rho_{f_1, g_{(\beta)_1}} \left(a_0 + \frac{a_1}{f_1^{\sharp m}} \right) = a_0 + \frac{a_1 \sharp r_{(\beta)_1}^{\sharp m}}{g_{(\beta)_1}^{\sharp m \theta_{(\beta)}}},$$

which implies that there exist $t', d', \delta'_{\beta_i} \in \mathcal{Z}_{>0}$, $\beta_i \in \Lambda_1$ and $h_{(\beta_i)_0} \in R_0$ such that

$$f_1^{t'} = \sum_{i=1}^{d'} h_{(\beta_i)_1} g_{(\beta_i)_1}^{\delta'_{\beta_i}} \quad (79)$$

and

$$a_0 = f_1^{\sharp t'} \sharp a_1 = 0. \quad (80)$$

Hence, $s = 0$ by (80).

This proves that the properties (i) in Definition 6.2 holds for the basic open set $D^\sharp(f_i)$ and the open covering (67).

In order to prove that the properties (ii) in Definition 6.2 holds for the basic open set $D^\sharp(f_i)$ and its open covering

$$\{ D^\sharp(g_{(\alpha)_0}) \mid \alpha \in \Lambda_0 \} \bigcup \{ D^\sharp(g_{(\beta)_1}) \mid \beta \in \Lambda_1 \}, \quad (81)$$

it is sufficient to prove that the properties (ii) in Definition 6.2 holds for the basic open set $D^\sharp(f_i)$ and a finite subcovering of the open covering given by (81). A direct computation proves that this fact is indeed true.

This completes the proof of Proposition 6.2. □

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